FIN 620
Emp. Methods in Finance

Lecture 1 – Linear Regression I

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Today's Agenda

- Introduction
- Discussion of Syllabus
- Review of linear regressions

My expectation is that you've seen most of this before; but it is helpful to review the key ideas that are useful in practice (without all the math)

Despite trying to do much of it without math; today's lecture likely to be long and tedious… (sorry)

Linear Regression – *Outline*

- The CEF and causality (very brief)
- Linear OLS model
- Multivariate estimation
- Hypothesis testing
- Miscellaneous issues

We will cover the latter two in the next lecture

Background readings

- Angrist and Pischke
  - *Sections 3.1-3.2, 3.4.1*
- Wooldridge
  - *Sections 4.1 & 4.2*
- Greene
  - *Chapter 3 and Sections 4.1-4.4, 5.7-5.9, 6.1-6.2*
- Cohn, Liu, Wardlaw (*JFE* 2022)

Motivation

- Linear regression is arguably the most popular modeling approach in corporate finance
  - Transparent and intuitive
  - Very robust technique; easy to build on
  - Even if not interested in causality, it is useful for describing the data

Given importance, we will spend today & next lecture reviewing the key ideas

Motivation continued…

- As researchers, we are interested explaining how the world works
  - E.g., how are firms' choices regarding leverage *explained* by their investment opportunities
    - I.e., if investment opportunities suddenly jumped for some random reason, how would we expect firms' leverage to respond on average?
  - More broadly, how is y explained by x, where both y and x are random variables?

Linear Regression – *Outline*

- The CEF and causality (very brief)
  - Random variables & the CEF
  - Using OLS to learn about the CEF
  - Briefly describe "causality"
- Linear OLS model
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A bit about random variables

- With this in mind, it is useful know that any random variable y can be written as

$$y = E(y \mid x) + \varepsilon$$

where (y, x, ε) are random variables and $E(\varepsilon \mid x)=0$

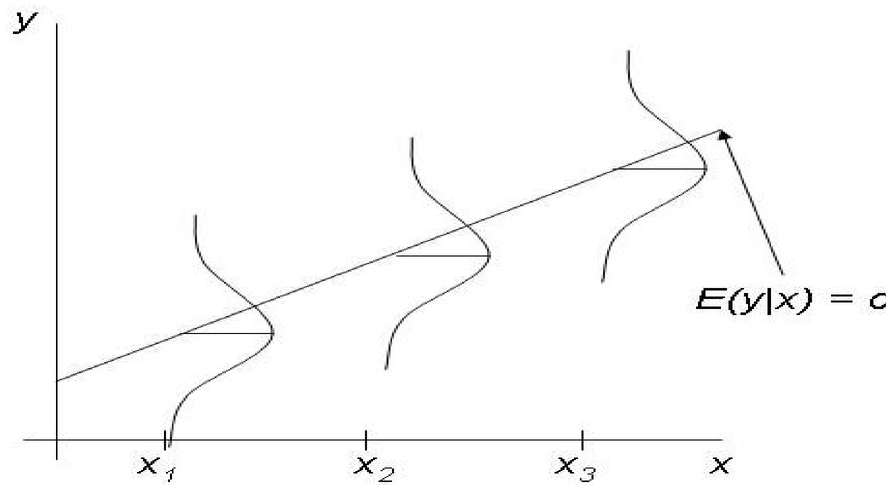
  - $E(y \mid x)$ is expected value of y given x
  - In words, y can be broken down into part 'explained' by x, $E(y \mid x)$, and a piece that is mean independent of x, ε

Conditional expectation function (CEF)

- $E(y|x)$ is what we call the CEF, and it has very desirable properties
  - Natural way to think about relationship between x and y
  - And it is <u>best predictor of y given x</u> in a minimum mean-squared error sense
    - I.e., $E(y|x)$ minimizes $E[(y\text{-}m(x))^2]$,where $m(x)$ can be <u>any</u> function of x.

CEF visually…

- $E(y|x)$ is fixed, but <u>unobservable</u>



Our goal is to learn about the CEF

- Intuition: for any value of x, distribution of y is centered about $E(y|x)$

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Linear regression and the CEF

- If done correctly, a linear regression can help us uncover what the CEF is
- Consider linear regression model, $y = \beta x + u$
  - y = dependent variable
  - x = independent variable
  - u = error term (or disturbance)
  - β = slope parameter

Some additional terminology

- Other terms for y…
  - Outcome variable
  - Response variable
  - Explained variable
  - Predicted variable
  - Regressand
- Other terms for x…
  - Covariate
  - Control variable
  - Explanatory variable
  - Predictor variable
  - Regressor

Details about $y = \beta x + u$

- (y, x, u) are random variables
- (y, x) are observable
- (u, β) are <u>un</u>observable
  - u captures everything that determines y after accounting for x *[This might be a lot of stuff!]*
  - We want to estimate β

Ordinary Least Squares (OLS)

- Simply put, OLS finds the β that minimizes the mean-squared error

$$\beta = \arg\min_{b} = E[(y - bx)^2]$$

- Using first order condition: $E[x(y-\beta x)]=0$, we have $\beta = E(xy)/E(x^2)$
- **Note:** by definition, the residual from this regression, $y-\beta x$, is uncorrelated with x

What great about this linear regression?

- It can be proved that…
  - βx is best* linear prediction of y given x
  - βx is best* linear approximation of $E(y|x)$

    * 'best' in terms of minimum mean-squared error

- This is quite useful. I.e., even if $E(y|x)$ is *nonlinear*, the regression gives us the best linear approximation of it

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What about causality?

- Need to be careful here…
  - How x explains y, which this regression helps us understand, is not the same as learning the causal effect of x on y
  - For that, we need more assumptions…

The basic assumptions *[Part 1]*

- *Assumption #1:* $E(u) = 0$
  - With intercept, this is totally innocuous
  - Just change regression to $y = \alpha + \beta x + u$, where α is the intercept term
  - Now suppose, $E(u)=k\neq 0$
    - We could rewrite $u = k + w$, where $E(w)=0$
    - Then, model becomes $y = (\alpha + k) + \beta x + w$
    - Intercept is now just $\alpha + k$, and error, w, is mean zero
    - I.e., Any non-zero mean is absorbed by intercept

The basic assumptions *[Part 2]*

Intuition?

- *Assumption #2:* $E(u|x) = E(u)$
  - In words, average of u (i.e., unexplained portion of y) does not depend on value of x
  - This is "conditional mean independence" (CMI)
    - True if x and u are independent of each other
    - Implies u and x are uncorrelated

This is the key assumption being made when people make causal inferences

CMI Assumption

- Basically, assumption says you've got correct CEF model for <u>causal</u> effect of x on y
  - CEF is <u>causal</u> if it describes differences in average outcomes for a change in x
    - i.e., change in y if x increases from values a to b is equal to $E(y|x=b)-E(y|x=a)$ **[In words?]**
  - Easy to see that this is only true if $E(u|x) = E(u)$ *[This is done on next slide…]*

Example of why CMI is needed

- With model $y = \alpha + \beta x + u$,
  - $E(y|x=a) = \alpha + \beta a + E(u|x=a)$
  - $E(y|x=b) = \alpha + \beta b + E(u|x=b)$
  - Thus, $E(y|x=b) - E(y|x=a) = \beta(b-a) + E(u|x=b) - E(u|x=a)$
  - This only equals what we think of as the 'causal' effect of x changing from a to b if $E(u|x=b) = E(u|x=a)$… i.e., CMI assumption holds

Tangent – CMI versus correlation

- CMI (which implies x and u are uncorrelated) is needed for <u>no bias</u> *[which is a finite sample property]*
- However, we only need to assume a zero correlation between x and u for <u>consistency</u> *[which is a large sample property]*
  - More about bias *vs.* consistency later; but we typically care about consistency, which is why I'll often refer to correlations rather than CMI

Is it plausible?

- Admittedly, there are many reasons why this assumption might be violated
  - Recall, u captures all the factors that affect y other than x… It will contain a lot!
  - Let's just do a couple of examples…

Ex. #1 – Capital structure regression

- Consider following firm-level regression:

$$Leverage_i = \alpha + \beta Profitability_i + u_i$$

  - CMI implies average *u* is same for each profitability
  - Easy to find a few stories why this isn't true…
    - #1 – unprofitable firms tend to have higher bankruptcy risk, which by tradeoff theory, should mean a <u>lower</u> leverage
    - #2 – unprofitable firms have accumulated less cash, which by pecking order means they should have <u>more</u> leverage

Ex. #2 – Investment

Measure of investment opportunities

- Consider following firm-level regression:

$$Investment_i = \alpha + \beta Q_i + u_i$$

  - CMI implies average *u* is same for each Tobin's Q
  - Easy to find a few stories why this isn't true…
    - #1 – Firms with low Q might be in distress & invest less
    - #2 – Firms with high Q might be smaller, younger firms that have a harder time raising capital to fund investments

Is there a way to test for CMI?

- Let $\hat{y}$ be the predicted value of y, i.e. $\hat{y} = \alpha + \beta x$, where α and β are OLS estimates
- And, let $\hat{u}$ be the residual, i.e. $\hat{u} = y - \hat{y}$
- Can we prove CMI if residuals if $E(\hat{u})=0$ and if $\hat{u}$ is uncorrelated with x?
  - **Answer:** No! <u>By construction</u>, these residuals are mean zero and uncorrelated with x. See earlier derivation of OLS estimates

Identification police

- What people call the “identification police” are those that look for violations of CMI
  - I.e., the “police” look for a reason why the model’s disturbance is correlated with x
    - Unfortunately, it’s not that hard…
    - **Trying to find ways to ensure the CMI assumption holds and causal inferences can be made will be a key focus of this course**

A side note about “endogeneity”

- Many “police” will criticize a model by saying it has an “endogeneity problem” but then don’t say anything further…
- **But what does it mean to say there is an “an endogeneity problem”?**

A side note about “endogeneity”

- **My view:** such vague “endogeneity” critics suspect something is potentially wrong, but don’t really know why or how
  - **Don’t let this be you! Be specific about what the problem is!**
- Violations to CMI can be roughly categorized into three bins… which are?

Three reasons why CMI is violated

- Omitted variable bias
- Measurement error bias
- Simultaneity bias
  - We will look at each of these in much more detail in the "Causality" lecture

What "endogenous" means to me

- An "endogenous" x is when its value depends on y (i.e., it determined jointly with y such that there is simultaneity bias).
  - However, some use a broader definition to mean any correlation between x and u *[e.g., Roberts & Whited (2011)]*
  - Because of the confusion, I avoid using "endogeneity"; I'd recommend the same for you
    - I.e., Be specific about CMI violation; e.g., just say omitted variable, measurement error, or simultaneity bias

A note about presentations…

- Think about "causality" when presenting next week and the following week
  - I haven't yet formalized the various reasons for why "causal" inferences shouldn't be made; but I'd like you to take a stab at thinking about it

Linear Regression – *Outline*

- The CEF and causality (very brief)
- Linear OLS model
  - Basic interpretation
  - Rescaling & shifting of variables
  - Incorporating non-linearities
- Multivariate estimation
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Interpreting the estimates

- Suppose I estimate the following model of CEO compensation

$$salary_i = \alpha + \beta ROE_i + u_i$$

  - Salary for CEO *i* is in \$000s; ROE is a %

- If you get… $\hat{\alpha} = 963.2$

$$\hat{\beta} = 18.50$$

  - What do these coefficients tell us?
  - Is CMI likely satisfied?

Interpreting the estimates – *Answers*

$$salary_i = 963.2 + 18.5 ROE_i + u_i$$

- What do these coefficients tell us?
  - 1 percentage point increase in ROE is associated with \$18,500 increase in salary
  - Average salary for CEO with ROE = 0 was equal to \$963,200
- Is CMI likely satisfied? **Probably not**

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Scaling the *dependent variable*

- What if I change measurement of salary from \$000s to \$s by multiplying it by 1,000?
  - Estimates were… $\hat{\alpha} = 963.2$
    $\hat{\beta} = 18.50$
  - Now, they will be… $\hat{\alpha} = 963,200$
    $\hat{\beta} = 18,500$

Scaling y continued…

- Scaling y by an amount c just causes all the estimates to be scaled by the same amount
  - Mathematically, easy to see why…

$$y = \alpha + \beta x + u$$

$$cy = \underbrace{(c\alpha)}_{\text{New intercept}} + \underbrace{(c\beta)}_{\text{New slope}} x + cu$$

Scaling *y* continued…

- Notice, the scaling has *no* effect on the relationship between ROE and salary
  - I.e., because *y* is expressed in \$s now, $\hat{\beta} = 18{,}500$ means that a one percentage point increase in ROE is still associated with \$18,500 increase in salary

Scaling the *independent variable*

- What if I instead change measurement of ROE from percentage to decimal? (i.e., multiply ROE by 1/100)
  - Estimates were… $\hat{\alpha} = 963.2$ $\hat{\beta} = 18.50$
  - Now, they will be… $\hat{\alpha} = 963.2$ $\hat{\beta} = 1{,}850$

Scaling x continued…

- Scaling x by an amount k just causes the slope on x to be scaled by $1/k$
  - Mathematically, easy to see why…

$$y = \alpha + \beta x + u$$

$$y = \alpha + \left(\frac{\beta}{k}\right) kx + u$$

New slope

Will interpretation of estimates change?

Answer: Again, no!

Scaling both *x* and *y*

- If scale y by an amount c and x by amount k , then we get…
  - Intercept scaled by c
  - Slope scaled by c/k

$$y = \alpha + \beta x + u$$

$$cy = (c\alpha) + \left(\frac{c\beta}{k}\right) kx + cu$$

- **When is scaling useful?**

Practical application of scaling **#1**

- No one wants to see a coefficient of 0.000000456 **or** 1,234,567,890
- Just scale the variables for cosmetic purposes!
  - It will affect coefficients & SEs
  - However, it won't affect *t*-stats or inference

Practical application of scaling **#2** *[P1]*

- To improve interpretation, in terms of found magnitudes, helpful to scale by the variables by their sample standard deviation
  - Let σ_x and σ_y be sample standard deviations of x and y respectively
  - Let c, the scalar for y, be equal to $1/\sigma_y$
  - Let k, the scalar for x, be equal to $1/\sigma_x$
  - I.e., unit of x and y is now standard deviations

Practical application of scaling **#2** *[P2]*

- With the prior rescaling, how would we interpret a slope coefficient of 0.25?
  - **Answer** = a 1 s.d. increase in x is associated with ¼ s.d. increase in y
  - The slope tells us how many standard deviations y changes, on average, for a standard deviation change in x
  - Is 0.25 large in magnitude? What about 0.01?

Shifting the variables

- Suppose we instead add c to y and k to x (i.e., we shift y and x up by c and k respectively)

- Will the estimated slope change?

Shifting continued…

- No! Only the estimated intercept will change
  - Mathematically, easy to see why…

$$y = \alpha + \beta x + u$$

$$y + c = \alpha + c + \beta x + u$$

$$y + c = \alpha + c + \beta\left(x + k\right) - \beta k + u$$

$$y + c = \left(\alpha + c - \beta k\right) + \beta\left(x + k\right) + u$$

New intercept

Slope the same

Practical application of shifting

- To improve interpretation, sometimes helpful to demean x by its sample mean
  - Let μ_x be the sample mean of x; regress y on x - μ_x
  - Intercept now reflects expected value of y for $x = \mu_x$

$$y = (\alpha + \beta\mu_x) + \beta(x - \mu_x) + u$$

$$E(y \mid x = \mu_x) = (\alpha + \beta\mu_x)$$

  - This will be very useful when we get to diff-in-diffs

Break Time

- Let's take a 10-minute break

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Incorporating nonlinearities *[Part 1]*

- Assuming that the causal CEF is linear may not always be that realistic
  - E.g., consider the following regression

$$\boldsymbol{wage = \alpha + \beta education + u}$$

  - Why might a linear relationship between # of years of education and level of wages be unrealistic? How can we fix it?

Incorporating nonlinearities *[Part 2]*

- Better assumption might be that each year of education leads to a constant proportionate (i.e., percentage) increase in wages
  - Approximation of this intuition captured by…

$$ln(wage) = \alpha + \beta education + u$$

  - I.e., the linear specification is very flexible because it can capture linear relationships between non-linear variables

Common nonlinear function forms

- Regressing Levels on Logs
- Regressing Logs on Levels
- Regressing Logs on Logs

Let’s discuss how to interpret each of these

The usefulness of log

- Log variables are useful because $100 \times \Delta \ln(y) \approx \%\Delta y$
  - **Note:** When I (and others) say "Log", we really mean the natural logarithm, "Ln". E.g., if you use the "log" function in Stata, it assumes you meant "ln"

Interpreting log-level regressions

- If estimate, the ln(wage) equation, 100β will tell you the %Δwage for an additional year of education. To see this…

$$\ln(wage) = \alpha + \beta education + u$$

$$\Delta \ln(wage) = \beta \Delta education$$

$$100 \times \Delta \ln(wage) = (100\beta)\Delta education$$

$$\%\Delta wage \approx (100\beta)\Delta education$$

Log-level interpretation continued…

- The proportionate change in y for a given change in x is assumed constant
  - The change in y is not assumed to be constant… it gets larger as x increases
  - Specifically, ln(y) is assumed to be linear in x; but y is <u>not</u> a linear function of x…

$$\ln(y) = \alpha + \beta x + u$$

$$y = \exp(\alpha + \beta x + u)$$

Example interpretation

- Suppose you estimated the wage equation (where wages are \$/hour) and got…

$$ln(wage) = 0.584 + 0.083education$$

  - What does an additional year of education get you?

Answer = 8.3% increase in wages.

  - Any potential problems with the specification?
  - Should we interpret the intercept?

Interpreting *log-log* regressions

- If estimate the following…

$$\ln(y) = \alpha + \beta \ln(x) + u$$

- β is the **elasticity** of *y* w.r.t. *x*!
  - i.e., β is the percentage change in *y* for a percentage change in *x*
  - **Note:** regression assumes constant elasticity between *y* and *x* regardless of level of *x*

Example interpretation of log-log

- Suppose you estimated the CEO salary model using logs got the following:

$$\ln(salary) = 4.822 + 0.257 \ln(sales)$$

- What is the interpretation of 0.257?

Answer = For each 1% increase in sales, salary increases by 0.257%

Interpreting *level-log* regressions

- If estimate the following…

$$y = \alpha + \beta \ln(x) + u$$

- β/100 is the change in *y* for 1% change *x*

Example interpretation of level-log

- Suppose you estimated the CEO salary model using logs got the following, where salary is expressed in \$000s:

$$\boldsymbol{salary = 4.822 + 1{,}812.5\,ln(sales)}$$

- What is the interpretation of 1,812.5?

  Answer = For each 1% increase in sales, salary increases by \$18,125

Summary of log functional forms

Model	Dependent Variable	Independent Variable	Interpretation of β
Level-Level	y	x	$dy = \beta dx$
Level-Log	y	ln(x)	$dy = (\beta / 100)\% dx$
Log-Level	ln(y)	x	$\% dy = (100\beta) dx$
Log-Log	ln(y)	ln(x)	$\% dy = \beta \% dx$

- Now, let's talking about what happens if you change units (i.e., scale) for either *y* or *x* in these regressions…

Rescaling logs doesn't matter *[Part 1]*

- What happens to intercept & slope if rescale (i.e., change units) of y when in log form?
- **Answer** = Only intercept changes; slope unaffected because it measures proportional change in y in Log-Level model

$$\log(y) = \alpha + \beta x + u$$
$$\log(c) + \log(y) = \log(c) + \alpha + \beta x + u$$
$$\log(cy) = \left(\log(c) + \alpha\right) + \beta x + u$$

Rescaling logs doesn't matter *[Part 2]*

- Same logic applies to changing scale of x in level-log models… only intercept changes

$$y = \alpha + \beta \log(x) + u$$

$$y + \beta \log(c) = \alpha + \beta \log(x) + \beta \log(c) + u$$

$$y = \left(\alpha - \beta \log(c)\right) + \beta \log(cx) + u$$

Rescaling logs doesn't matter *[Part 3]*

- **Basic message** – If you rescale a logged variable, it will not affect the slope coefficient because you are only looking at proportionate changes

Log approximation problems

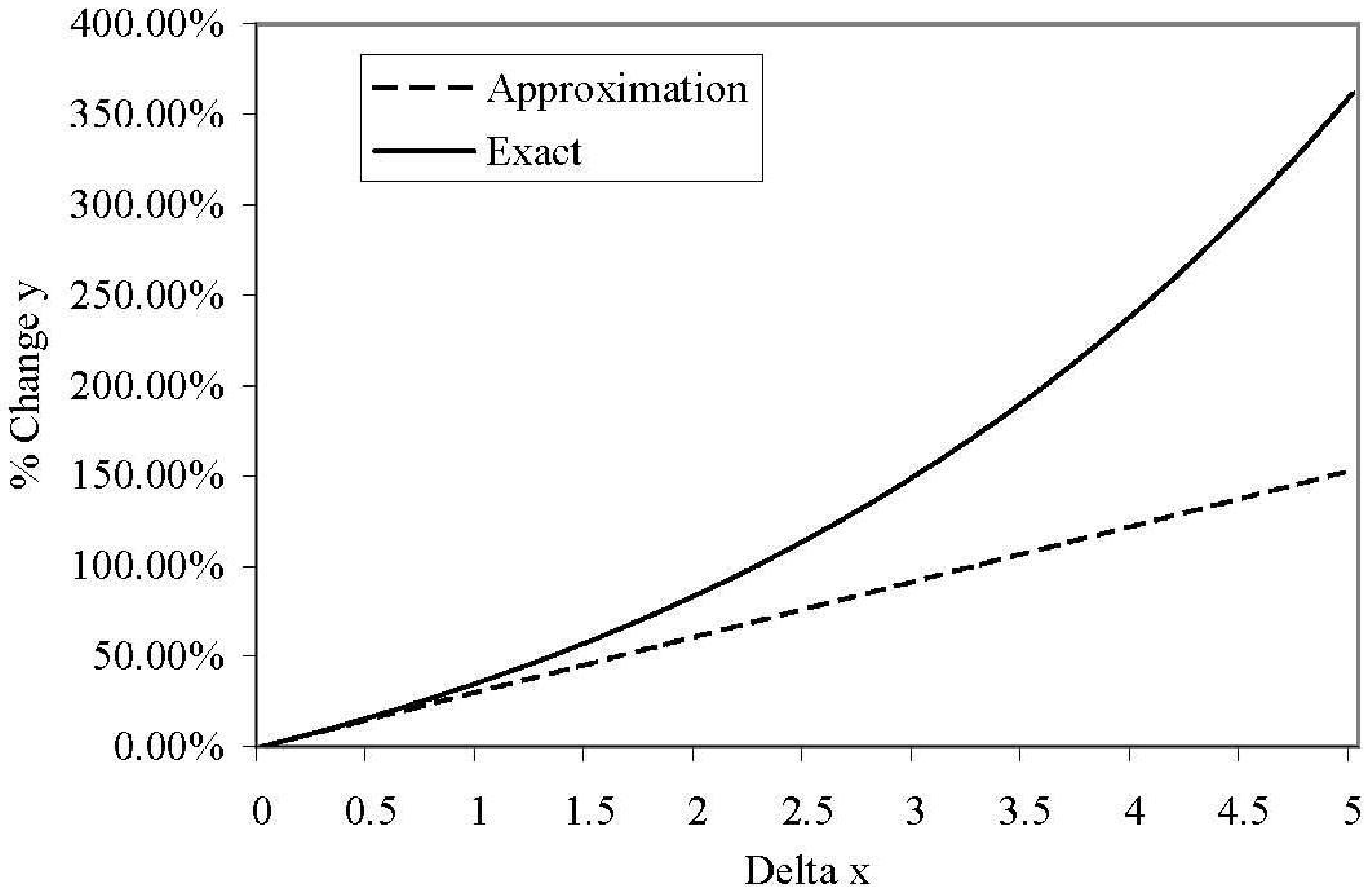
- I once discussed a paper where author argued that allowing capital inflows into country caused -120% change in stock prices during crisis periods…
  - **Do you see a problem with this?**
    - Of course! A 120% drop in stock prices isn't possible. The true percentage change was -70%. Here is where that author went wrong…

Log approximation problems *[Part 1]*

- Approximation error occurs because as true %Δy becomes larger, 100Δln(y)≈%Δy becomes a worse approximation

- To see this, consider a change from y to y'…
  - **Ex. #1:** $\dfrac{y'-y}{y} = 5\%$, and 100Δln(y) = 4.9%
  - **Ex. #2:** $\dfrac{y'-y}{y} = 75\%$, <u>but</u> 100Δln(y)= 56%

Log approximation problems *[Part 2]*



Log approximation problems *[Part 3]*

- Problem also occurs for negative changes
  - **Ex. #1:** $\frac{y'-y}{y} = -5\%$, and 100Δln(y) = -5.1%
  - **Ex. #2:** $\frac{y'-y}{y} = -75\%$, <u>but</u> 100Δln(y)= -139%

Log approximation problems *[Part 4]*

- So, if implied percent change is large, better to convert it to true % change before interpreting the estimate

$$\ln(y) = \alpha + \beta x + u$$

$$\ln(y') - \ln(y) = \beta(x' - x)$$

$$\ln(y'/y) = \beta(x' - x)$$

$$y'/y = \exp\left(\beta(x' - x)\right)$$

$$\left[(y' - y)/y\right]\% = 100\left[\exp\left(\beta(x' - x)\right) - 1\right]$$

Log approximation problems *[Part 5]*

- We can now use this formula to see what true % change in y is for $x' - x = 1$

$$[(y' - y)/y]\% = 100\left[\exp\left(\beta(x' - x)\right) - 1\right]$$

$$[(y' - y)/y]\% = 100\left[\exp\left(\beta\right) - 1\right]$$

  - If $\beta = 0.56$, the percent change isn't 56%, it is

$$100\left[\exp\left(0.56\right) - 1\right] = 75\%$$

Recap of last two points on logs

- Two things to keep in mind about using logs
  - Rescaling a logged variable doesn't affect slope coefficients; it will only affect intercept
  - Log is only approximation for % change; it can be a very bad approximation for large changes

Usefulness of logs – **Summary**

- Using logs gives coefficients with appealing interpretation
- Can be ignorant about unit of measurement of log variables since they're proportionate Δs
- Logs of y or x can mitigate influence of outliers

“Rules of thumb” on when to use logs

- Helpful to take logs for variables with…
  - Positive currency amount
  - Large integral values (e.g., population)
- Don’t take logs for variables measured in years or for variables that can equal zero…

What about using ln(1+y)?

- Because ln(0) doesn't exist, some use ln(1+*y*) for non-negative variables, i.e. $y \in [0, \infty)$
  - However, you should not do this! Nice interpretation no longer true, especially if a lot of zeros or many small values in *y* ***[Why?]***
    - ***Ex. #1:*** *What does it mean to go from ln(0) to ln(x>0)?*
    - ***Ex. #2:*** *And Ln(x'+1) – Ln(x+1) is not percent change of x*
  - See Cohn, Liu, Wardlaw (*JFE* 2022) for solutions & more details on why using ln(1+*y*) is problematic

Tangent – Percentage Change

- What is the percent change in unemployment if it goes from 10% to 9%?
  - This is 10 percent drop
  - It is a 1-percentage point drop
    - **Percentage change** is $[(x_1 - x_0)/x_0] \times 100$
    - **Percentage point change** is the raw change in percentages

Please take care to get this right in description of your empirical results

Models with quadratic terms *[Part 1]*

- Consider $y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$
- Partial effect of x is given by…

$$\Delta y = \left(\beta_1 + 2\beta_2 x\right)\Delta x$$

  - What is different about this partial effect relative to everything we've seen thus far?
    - **Answer** = It depends on the value of x. So, we will need to pick a value of x to evaluation (e.g. $\bar{x}$)

Models with quadratic terms *[Part 2]*

- If $\hat{\beta}_1 > 0, \hat{\beta}_2 < 0$, then it has parabolic relation
  - Turning point = Maximum = $\left|\hat{\beta}_1 / 2\hat{\beta}_2\right|$
  - *Know where this turning point is!* Don't claim a parabolic relation if it lies outside range of *x*!
  - Odd values might imply misspecification or simply mean the quadratic terms are irrelevant and should be excluded from the regression

Linear Regression – *Outline*

- The CEF and causality (very brief)
- Linear OLS model
- Multivariate estimation
  - Properties & Interpretation
  - Partial regression interpretation
  - R^2, bias, and consistency
- Hypothesis testing
- Miscellaneous issues

Motivation

- Rather uncommon that we have just one independent variable
  - So, now we will look at multivariate OLS models and their properties…

Basic multivariable model

- Example with constant and k regressors

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + u$$

- Similar identifying assumptions as before
  - No collinearity among covariates **[why?]**
  - $E(u \mid x_1, \ldots, x_k) = 0$
    - Implies no correlation between <u>any</u> x and u, which means we have the correct model of the true causal relationship between y and $(x_1, \ldots, x_k)$

Interpretation of estimates

- Estimated intercept, $\hat{\beta}_0$, is predicted value of y when all $x = 0$; sometimes this makes sense, sometimes it doesn't
- Estimated slopes, $\left(\hat{\beta}_1, \ldots, \hat{\beta}_k\right)$, have a more subtle interpretation now…

$$y = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \ldots + \hat{\beta}_k x_k + \hat{u}$$

  - How would you interpret $\hat{\beta}_1$?

Interpretation – *Answer*

- Estimated slopes, $\left(\hat{\beta}_1, \ldots, \hat{\beta}_k\right)$, have partial effect interpretations
- Typically, we think about change in just one variable, e.g., $\Delta\, x_1$, holding constant all other variables, i.e., ($\Delta x_2, \ldots, \Delta x_k$ all equal 0)
  - This is given by $\Delta\hat{y} = \hat{\beta}_1 \Delta x_1$
  - I.e., $\hat{\beta}_1$ is the coefficient holding *all else fixed* (ceteris paribus)

Interpretation continued…

- However, can also look at how changes in multiple variables at once affects predicted value of y
  - I.e., given changes in x_1 through x_k we obtain the predicted change in y, Δy

$$\Delta\hat{y} = \hat{\beta}_1 \Delta x_1 + \ldots + \hat{\beta}_k \Delta x_k$$

Example interpretation – College GPA

- Suppose we regress college GPA onto high school GPA (4-point scale) and ACT score for N = 141 university students

$$colGPA = 1.29 + 0.453 hsGPA + 0.0094 ACT$$

  - **What does the intercept tell us?**
  - **What does the slope on *hsGPA* tell us?**

Example – Answers

- Intercept meaningless… person with zero high school GPA and ACT doesn't exist
- Example interpretation of slope…
  - Consider two students, Ann and Bob, with identical ACT score, but Ann's GPA is 1 point higher than Bob. Best prediction of Ann's college GPA is that it will be 0.453 higher than Bob's

Example continued…

- Now, what is effect of increasing high school GPA by 1 point and ACT by 1 point?

$$\Delta colGPA = 0.453 \times \Delta hsGPA + 0.0094 \times \Delta ACT$$

$$\Delta colGPA = 0.453 + 0.0094$$

$$\Delta colGPA = 0.4624$$

Example continued…

- Lastly, what is effect of increasing high school GPA by 2 points and ACT by 10 points?

$$\Delta colGPA = 0.453 \times \Delta hsGPA + 0.0094 \times \Delta ACT$$

$$\Delta colGPA = 0.453 \times 2 + 0.0094 \times 10$$

$$\Delta colGPA = 1$$

Fitted values and residuals

- Definition of residual for observation *i*, $\hat{u}_i$

$$\hat{u}_i = y_i - \hat{y}_i$$

- Properties of residual and fitted values
  - Sample average of residuals = 0; implies that sample average of $\hat{y}$ equals sample average of y
  - Sample covariance between each independent variable and residuals = 0
  - Point of means $(\bar{y}, \bar{x}_1, \ldots, \bar{x}_k)$ lies on regression line

Tangent about residuals

- Again, it bears repeating…
  - Looking at whether the residuals are correlated with the x's is NOT a test for causality
  - By construction, they are uncorrelated with x
  - There is no "test" of whether the CEF is the causal CEF; that justification will need to rely on <u>economic</u> arguments

Linear Regression – *Outline*

- The CEF and causality (very brief)
- Linear OLS model
- Multivariate estimation
  - Properties & Interpretation
  - Partial regression interpretation
  - R^2, bias, and consistency
- Hypothesis testing
- Miscellaneous issues

Question to motivate the topic…

- **What is wrong with the following? And why?**
  - Researcher wants to know effect of x on y after controlling for z
  - So, researcher removes the variation in y that is driven by z by regressing y on z & saves residuals
  - Then, researcher regresses these residuals on x and claims to have identified effect of x on y controlling for z using this regression

We'll answer why it's wrong in a second…

Partial regression *[Part 1]*

- The following is quite useful to know…
- Suppose you want to estimate the following

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

  - Is there another way to get $\hat{\beta}_1$ that doesn't involve estimating this directly?
    - **Answer:** Yes! You can estimate it by regressing the residuals from a regression of y on x_2 onto the residuals from a regression of x_1 onto x_2

Partial regression *[Part 2]*

- To be clear, you get $\hat{\beta}_1$, by…

  #1 – Regress y on x_2; save residuals (call them $\tilde{y}$)

  #2 – Regress x_1 on x_2; save residuals (call them $\tilde{x}$)

  #3 – Regress $\tilde{y}$ onto $\tilde{x}$; the estimated coefficient will be the same as if you'd just run the original multivariate regression!!!

Partial regression – *Interpretation*

- Multivariate estimation is basically finding effect of each independent variable after partialing out effect of other variables
  - I.e., Effect of x_1 on y after controlling for x_2, (i.e., what you'd get from regressing y on both x_1 and x_2) is the same as what you get after you partial out the effect x_2 from both x_1 and y and then run a regression using the residuals

Partial regression – *Generalized*

- This property holds more generally…
  - Suppose X_1 is vector of independent variables
  - X_2 is vector of more independent variables
  - And, you want to know that coefficients on X_1 that you would get from a multivariate regression of *y* onto all the variables in X_1 and X_2…

Partial regression – *Generalized, Part 2*

- You can get the coefficients for each variable in X_1 by…
  - Regress *y* and each variable in X_1 onto all the variables in X_2 (at once), save residuals from each regression
  - Do a regression of residuals; i.e., regress *y* onto variables of X_1, but replace *y* and X_1 with the residuals from the corresponding regression in step #1

Practical application of partial regression

- **Now, what is wrong with the following?**
  - Researcher wants to know effect of x on y after controlling for z
  - So, researcher removes the variation in y that is driven by z by regressing y on z & saves residuals
  - Then, researcher regresses these residuals on x and claims to have identified effect of x on y controlling for z using this regression

Practical application – Answer

- It's wrong because it didn't partial the effect of z out of x! Therefore, it is NOT the same as regressing y onto both x and z!
- <u>Unfortunately</u>, it was commonly done by researchers in finance *[e.g., industry-adjusting]*
  - We will see how badly this can mess up things in a later lecture where we look at my paper with David Matsa on unobserved heterogeneity

Linear Regression – *Outline*

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  - Partial regression interpretation
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- Miscellaneous issues

Goodness-of-Fit (R^2)

- A lot is made of R-squared; so, let's quickly review exactly what it is
- Start by defining the following:
  - Sum of squares total (SST)
  - Sum of squares explained (SSE)
  - Sum of squares residual (SSR)

Definition of SST, SSE, SST

If N is the number of observations and the regression has a constant, then

$$SST = \sum_{i=1}^{N} \left(y_i - \bar{y}\right)^2$$

SST is total variation in y

$$SSE = \sum_{i=1}^{N} \left(\hat{y}_i - \bar{y}\right)^2$$

SSE is total variation in predicted y
[mean of predicted y = mean of y]

$$SSR = \sum_{i=1}^{N} \hat{u}_i^2$$

SSR is total variation in residuals
[mean of residual = 0]

SSR, SST, and SSE continued…

- The total variation, SST, can be broken into two pieces… the explained part, SSE and unexplained part, SSR

$$\mathbf{SST = SSE + SSR}$$

- R^2 is just the share of total variation that is explained! In other words,

$$\mathbf{R^2 = SSE/SST = 1 - SSR/SST}$$

More about R^2

- As seen on last slide, R^2 must be between 0 and 1
- It can also be shown that R^2 is equal to the square of the correlation between *y* and predicted *y*
- If you add an independent variable, R^2 will never go down

Adjusted R^2

- Because R^2 always goes up, we often use what is called Adjusted R^2

$$AdjR^2 = 1 - \left(1 - R^2\right)\left(\frac{N-1}{N-1-k}\right)$$

  - k = # of regressors, excluding the constant
  - Basically, you get penalized for each additional regressor, such that adjusted R^2 won't go up after you add another variable if it doesn't improve fit much [it can go down!]

Interpreting R^2

- If I tell you the R^2 is 0.014 from a regression, what does that mean? Is it bad?
  - **Answer #1** = It means I'm only explaining about 1.4% of the variation in *y* with the regressors that I'm including in the regression
  - **Answer #2** = Not necessarily! It doesn't mean the model is wrong; you might still be getting a <u>consistent</u> estimate of the β you care about!

Unbiasedness *versus* Consistency

- When we say an estimate is unbiased or consistent, it means we think it has a causal interpretation…
  - I.e., the CMI assumption holds and the x's are all uncorrelated with the disturbance, u
- **Bias** refers to finite sample property; **consistency** refers to asymptotic property

More formally…

- An estimate, $\hat{\beta}$, is unbiased if $E\left(\hat{\beta}\right) = \beta$
  - I.e., on average, the estimate is centered around the true, unobserved value of β
  - Doesn't say whether you get a more precise estimate as sample size increases
- An estimate is consistent if $\underset{N \to \infty}{plim}\, \hat{\beta} = \beta$
  - I.e., as sample size increases, the estimate converges (in probability limit) to the true coefficient

Unbiasedness of OLS

- OLS will be unbiased when…
  - Model is linear in parameters
  - We have a random sample of x
  - No perfect collinearity between x's
  - $E(u|x_1,\ldots, x_k) = 0$
    [Earlier assumptions #1 and #2 give us this]
- Unbiasedness is nice feature of OLS; but in practice, we care more about consistency

Consistency of OLS

- OLS will be consistent when
  - Model is linear in parameters
  - *u* is not correlated with any of the *x*'s,

    [CMI assumptions #1 and #2 give us this]
- Again, this is good
- See textbooks for more information

Summary of Today *[Part 1]*

- The CEF, $E(y|x)$ has desirable properties
  - Linear OLS gives best linear approx. of it
  - If correlation between error, u, and independent variables, x's, is zero it has causal interpretation
- Scaling & shifting of variables doesn't affect inference, but can be useful
  - E.g., demean to give intercepts more meaningful interpretation or rescale for cosmetic purposes

Summary of Today *[Part 2]*

- Multivariate estimates are partial effects
  - I.e., effect of x_1 holding $x_2, \ldots, x_k$ constant
  - Can get same estimates in two steps by first partialing out some variables and regressing residuals on residuals in second step

Assign papers for next week…

- Angrist (AER 1990)
  - Military service & future earnings
- Angrist and Lavy (QJE 1999)
  - Class size & student achievements
- Acemoglu, et al. (AER 2001)
  - Institutions and economic development

These are seminal papers in economics with clever identification strategies… i.e., what we aspire to learn about later in the course

In First Half of Next Class

- Finish discussion of the linear regression
  - Hypothesis testing
  - Irrelevant regressors & multicollinearity
  - Binary variables & interactions
- Relevant readings; see syllabus